A BODY-FITTED CONFORMAL MAPPING METHOD WITH GRID-SPACING CONTROL

J. C. Wu

and

U. Gulcat

Georgia Institue of Technology

It is demonstrated by analyses and by numerical illustrations that any arbitrarily prescribed contour, open or closed, can be mapped conformally onto a simple contour, such as a unit circle, using any arbitrarily prescribed distribution of scale factor of transformation. This flexibility of selecting a scale factor distribution on the contour is not in violation of the well-known Riemann's uniqueness theory for conformal mapping. The much used Joukowski transformation is shown to be one of a family of conformal transformations that map a given airfoil contour onto a unit circle. For flow problems, the conformal mapping of a region bounded by a complicated contour onto a corresponding region bounded by a simple contour is of interest. With an arbitrarily prescribed scale factor, there exist in general singular points located at finite distances from the contour. (The case where singularities are located infinitely far from the contour is an exception.) Numerical methods for generating conformal grids should therefore incorporate a mechanism that ensures the absence of singular points in the region of interest. In this context, the distribution of scale factor on the contour cannot be arbitrary. The restriction on the scale factor distribution is not stringent. There exists ample freedom in the control of grid spacing on the contour so that, in general, the physics of the flow problem can be accommodated by a suitably designed conformal grid.

Preceding page blank

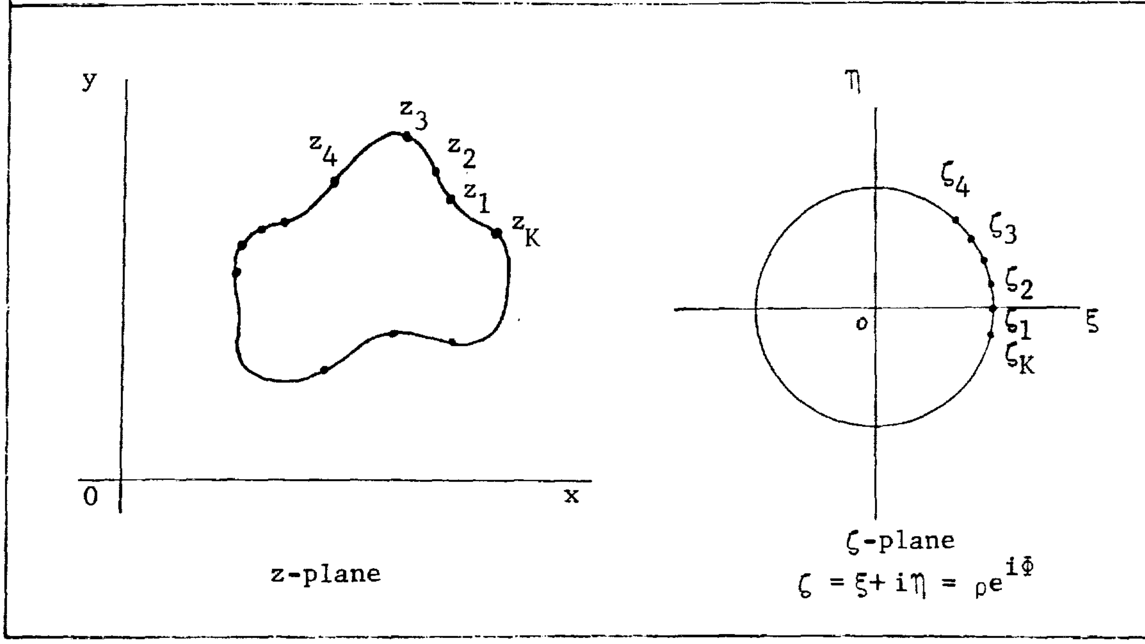
DESIRED FEATURES OF GRID SYSTEMS
• Be body-fitted. • Possess control over grid-spacing. • Yield algebraic equations amenable to highly efficient numerical procedures. • Require minimal computational efforts to generate.

The first feature listed above is generally accepted as being the key to the successful computation of flows. The second feature is essential to the computation of complex flows with diverse length scales in different regions of the flows. The third feature is critical in situations where the amount of computation required is very large. The fourth feature is important if repeated generation of grids is desired during the solution of a given problem. (For example, in the solution of a time-dependent problem, different grids may be desired for different time intervals).

CURVILINEAR COORDINATES AND GRID SYSTEM
• Non-orthogonal coordinates yield transformed differential equations that are substantially more complicated than the original equations.   • Orthogonal non-conformal coordinates yield less complicated equations.   • Conformal coordinates yield simplest transformed equations.   • The requirements that a transformation be conformal and that it possesses a grid-spacing-control ability are not mutually exclusive.   • Conformal mapping can be generated very efficiently.   • Orthogonal grids can be easily developed using conformal mapping.

The advantages of using conformal grids are most clearly demonstrated by the numerical procedures available for the Poisson's equation. Algebraic equations obtained in conformal grids can be solved using direct methods such as the block Gaussian elimination, the odd-even reduction, and the Fourier series methods. The choice of methods is somewhat more limited in an orthogonal non-conformal grid. With non-orthogonal grids, iterative procedures are generally required. The main purpose of this paper is to show that any prescribed two-dimensional body contour can be conformally mapped onto a simple shape, such as the unit circle, and such mappings do possess a grid-spacing-control ability.

CONFORMAL MAPPING



Any usual contour, open or closed, can be mapped conformally onto a simple contour, such as a circle or a straight line segment, using any prescribed distribution of the scale factor of transformation on the contour. This is true for smooth contours as well as for contours with discontinuous slopes. The unit circle is used as the canonical contour for the following discussion. A total of K equally spaced points are assigned on the unit circle, with the point ζ_k given by

$$\zeta_k = e^{iz\pi k/K}$$

where K is an odd integer.

The corresponding points, z_k, on the original contour are sequenced as shown but otherwise arbitrarily located.

THE LAURENT SERIES

$$\zeta = \rho\, e^{i\varphi}$$

$$\zeta_k = \exp\left(\frac{iz\pi k}{K}\right) \quad , \quad K \text{ an odd integer}$$

$$z = f(\zeta) = \sum_{n=-N}^{N} C_n \zeta^n \quad , \quad N = \frac{K-1}{2}$$

$$z_k = \sum_{n=-N}^{N} C_n \exp\left\{\frac{iz\pi kn}{K}\right\}$$

$$\sum_{k=-N}^{N} z_k \exp\left\{\frac{iz\pi km}{K}\right\} = \sum_{n=-N}^{N} C_n \sum_{n=-N}^{N} \exp\left\{\frac{iz\pi k(m-n)}{K}\right\}$$

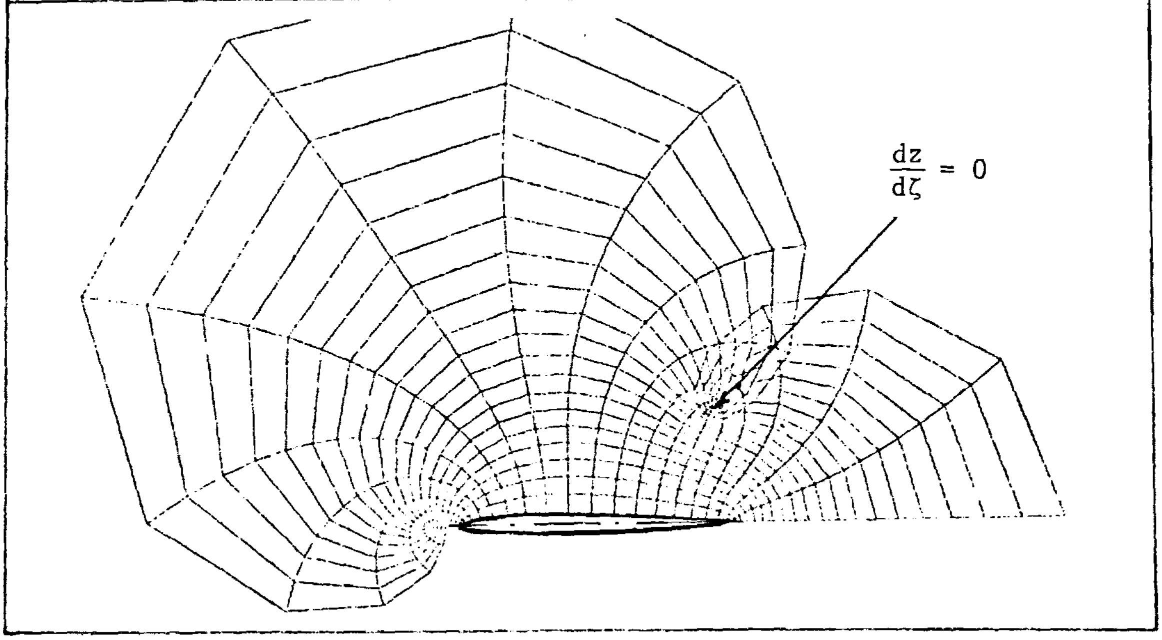
$$= \begin{cases} K & m=n \\ 0 & m=n \end{cases}$$

$$C_n = \frac{1}{K} \sum_{k=-N}^{N} z_k \exp\left\{\frac{iz\pi kn}{K}\right\}$$

$$z = \frac{1}{K} \sum_{n=-N}^{N} \sum_{k=-N}^{N} z_k \exp\left\{\frac{iz\pi kn}{K}\right\} \zeta^n$$

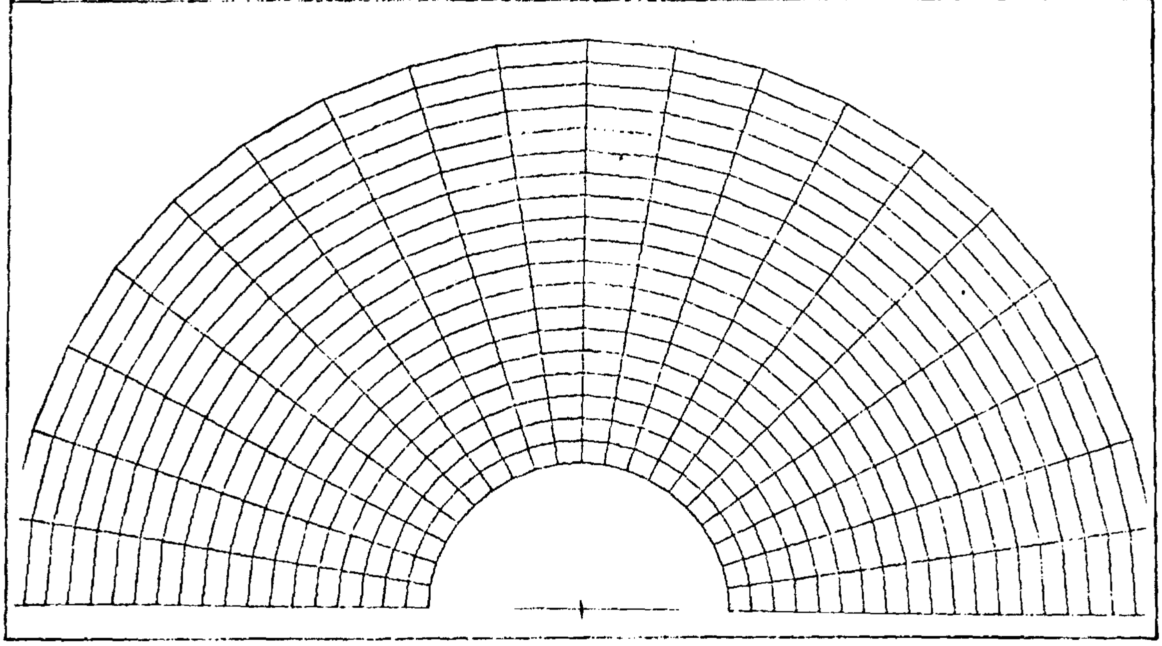
By analytic continuation, the Fourier coefficients of the Laurent series are those obtained above. The finite Laurent series therefore can be used to compute the grid-point locations away from the contour that corresponds to specified grid points in the ζ-plane. The above analysis can be carried out for an infinite Laurent series. The only change is that the Fourier coefficients are then expressed as integrals instead of sums. The finite Laurent series represents an approximation of the infinite Laurent series whose regular part converges inside a certain circle and whose principal part converges outside another certain circle. The domain of convergence of the infinite Laurent series is the common annulus of the two circles. The finite Laurent series produces accurate conformal grids in this domain of convergence. The conformality of the grids thus generated is ensured by the analyticity of the Laurent series.

A CONFORMAL GRID WITH SINGULAR POINTS
NEAR THE CONTOUR



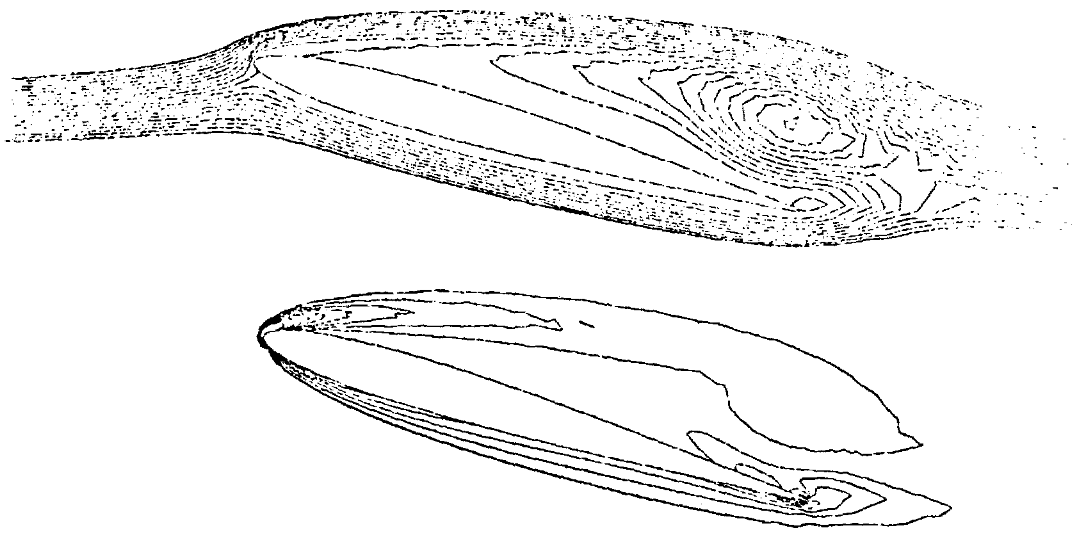
With an arbitrarily prescribed distribution of the scale factor, there exist in general "singular points" located at finite-distances from the contour. Therefore, numerical methods for generating conformal grids should contain provisions that ensure the absence of singular points in the region of computational interest. In this context, the distribution of the scale factor on the contour cannot be arbitrary. In this figure is shown a grid around a symmetric airfoil with singular points located near the airfoil. This figure is obtained using the finite Laurent series method. The prescribed points on the airfoil are symmetrically distributed about the line of symmetry of the airfoil. The grid lines shown are mapped onto the radial lines and concentric circles shown on the next figure.

THE GRID SYSTEM IN THE CIRCLE-PLANE



The "canonical" domain used here is the domain exterior to the unit circle. The grid lines shown here are mapped conformally onto the grid lines shown in the airfoil-planes at all points except the singular points where the mapping ceases to be conformal. The gird shown is orthogonal with equal spacings in the angular and the radial directions.

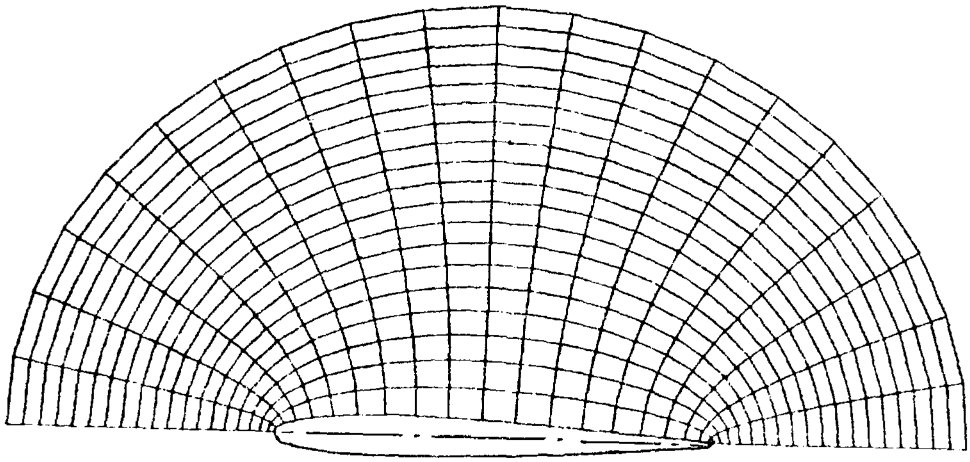
SOLUTION FIELD AROUND AN AIRFOIL



12% Thick Joukowski Airfoil
Re = 3.63×10^6, $\alpha = 15^\circ$, t = 13.2

Here are shown some streamlines and constant vorticity lines around a 9% thick symmetric airfoil. With the integro-differential approach the authors are using, the solution field can be confined to the vortical region of the flow. In consequence, the presence of the singular points in the conformal grid outside the vortical region is acceptable. With prevailing method, the flowfield is usually truncated and it is permissible to have singular points present outside the truncated region.

JOUKOWSKI GRID

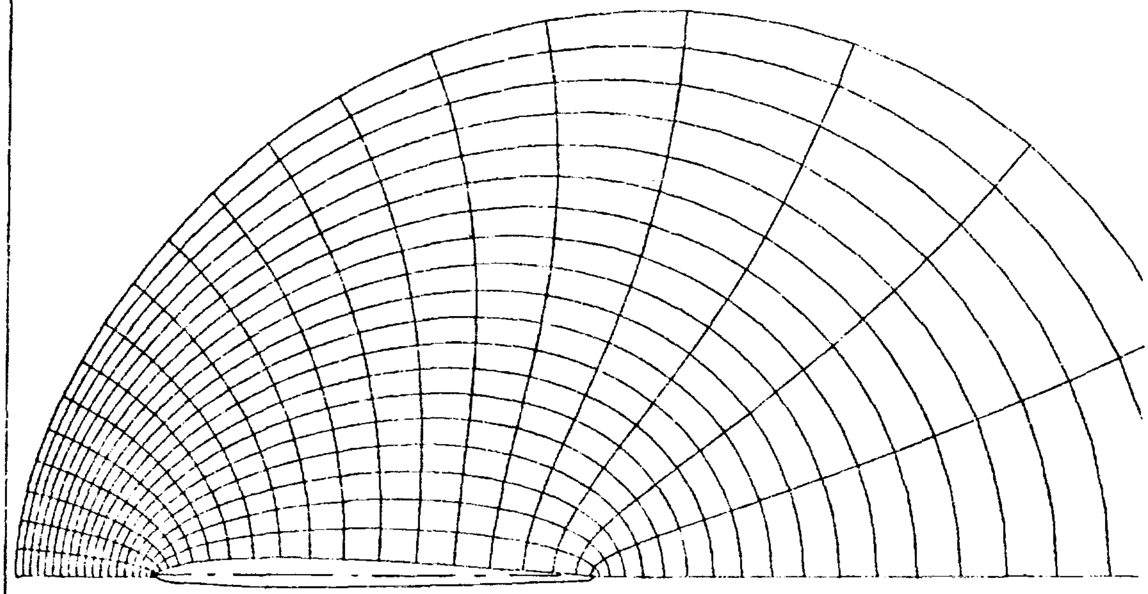


This figure shows the grid lines around a 9 % thick symmetric airfoil that is mapped using the Joukowski transformation

$$z = \zeta - 0.05214 + \frac{0.854078}{\zeta - 0.05214}$$

With this transformation there is no singular point at a finite distance from the airfoil. The trailing edge in this transformation is rounded (so as to avoid the need of the Schwarz-Christoffel procedure, which would have introduced complications unnecessary at this stage of development). The finite Laurent series method, with grid points on the airfoil boundary assigned properly, produces a grid system that is indistinguishable from the one shown.

A SYMMETRIC BILINEAR GRID

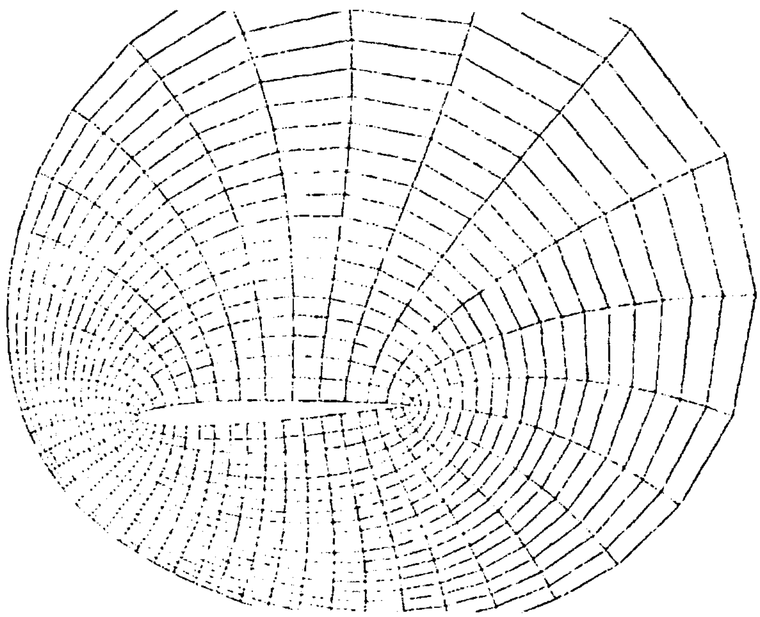


The unit circle in the ζ -plane can be mapped onto a unit circle in the w-plane through a bilinear transformation of the form

$$\zeta = \frac{w - \alpha}{1 - \alpha w}$$

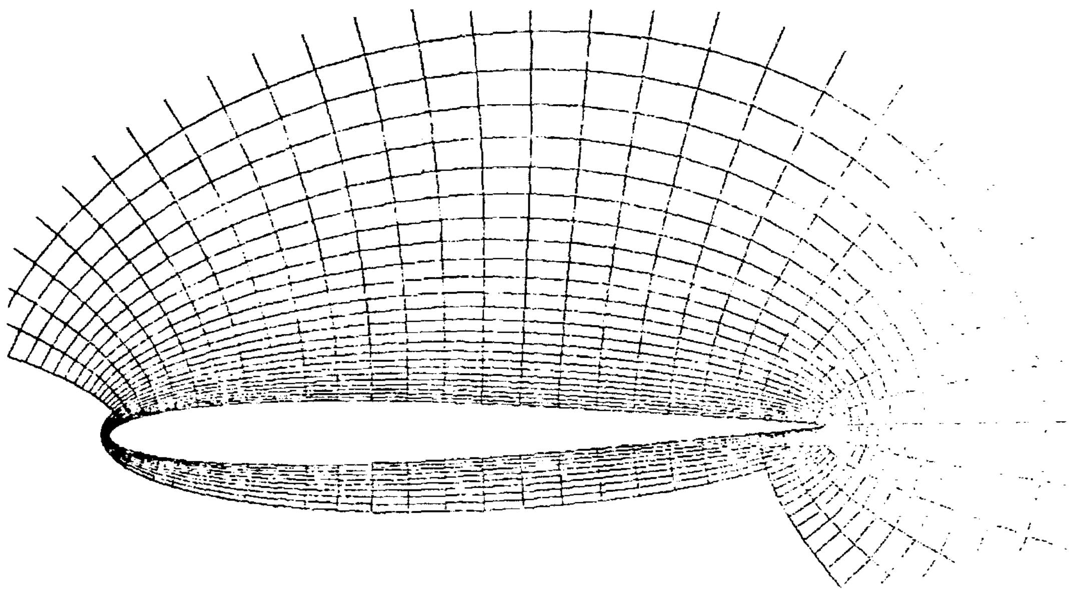
For any assigned value of α , the points on the airfoil boundary that correspond to uniformly distributed grid points on the unit circle can be located. The finite Laurent series method then yields a conformal mapping of a region exterior of the circle in the w-plane onto a region exterior of the airfoil in the z-plane. The concentric circles and radial lines in the w-plane are mapped onto the grid lines shown above for the case $\alpha = 1/8$. The grid lines are symmetric about the line of symmetry of the airfoil.

A NON-SYMMETRIC BILINEAR GRID



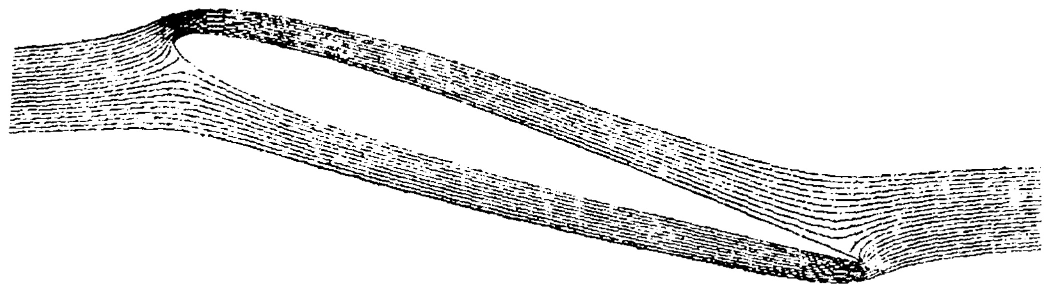
Using a complex value for α , the concentric circles and radial lines in the w-plane are mapped onto non-symmetric grid lines in the airfoil-plane. The figure above shows grid lines for the case $\alpha = \frac{1+i}{4\sqrt{2}}$. The singular points of the grid system shown here and in the previous figure are sufficiently far from the airfoil so that the grids are of practical interest.

A GRID IN USE



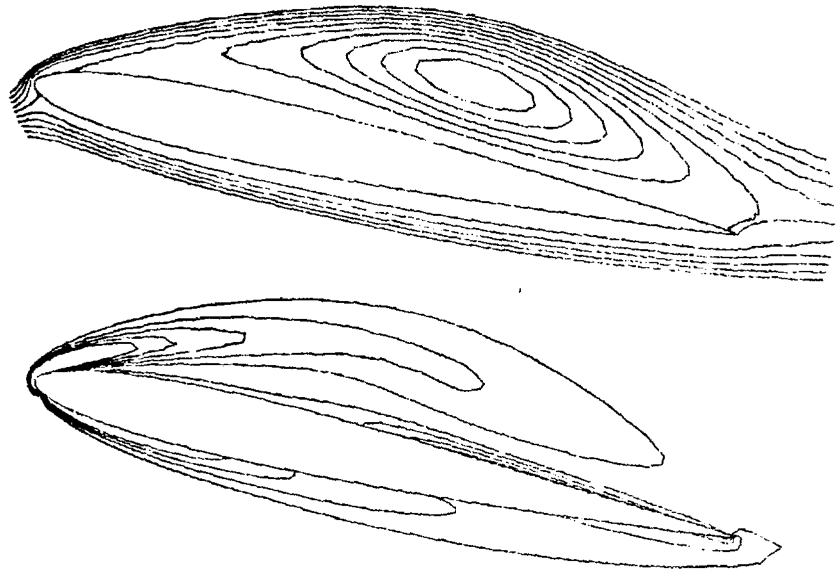
This grid has been used in a computation of a flow past a 9 % thick symmetric airfoil at an angle of attack of 15%. The grid is a bilinear grid with α = 16. In this study, the boundary layer region of the flow is computed separately from the detached region. It is only necessary to generate a grid covering the computation field and to keep the singular point away from this computation field.

INITIAL FLOW PATTERN AROUND AN AIRFOIL



This and the following figures show computed streamlines and vorticity contours around a 9 % thick airfoil set into motion impulsively and thereafter kept moving at a constant velocity with an angle of attack of 15° and a Reynolds number of 1000. This figure is for the time level immediately after the motion's onset. The vorticity is confined to the boundary of the airfoil and the flow away from the airfoil is potential. Note that the rear stagnation point is on the upper surface of the airfoil.

FLOW AROUND AN AIRFOIL WITH A SEPARATION BUBBLE



This figure shows the computed streamlines and constant vorticity contours around the airfoil after the airfoil has advanced 2.9 chord lengths relative to the freestream. A separation bubble has appeared and grown to its present size. The vorticity field is still confined to the region near the airfoil as shown. With the integro-differential approach used here, it is only necessary to perform computations in the vortical region. Therefore the grid needs only be generated for the vortical region.

OUTLINE OF GRID GENERATION PROCEDURE

The recommended procedure for generating a conformal grid with control over grid spacing consists of four steps:

(1) The locations of grid points on the physical contour that are mapped onto equally spaced points on a unit circle through a "Joukowski type" conformal transformation are computed.

(2) The coefficients in a finite Laurent series are computed as described earlier.

(3) A suitable bilinear transformation is introduced.

(4) Grid locations corresponding to concentric circles and radial lines in the bilinear transformed plane are computed.

A computer program (prepared by N. L. Sankar) which performs step (1) is available. This program uses an iterative procedure (Bauer et al, 1977 and other researchers). A spline approximation is utilized to achieve a high degree of accuracy. The operation count for this step is small. For each given contour, if several different grids are to be generated, then step (4) is the only step that needs to be repeated. Steps (1) and (2) need to be performed only once for the contour. Step (3) needs to be performed only once for all contours of interest.

CONCLUDING REMARKS

- Body-fitted conformal grids can be generated efficiently using the approach described.
- Ample freedom exists in the control of grid spacing on any contour so that the physics of the flow can be suitably accomadated.

The work reviewed here represents only the initial stage of development of a new conformal mapping approach for grid generation. Based on the results obtained thus far, this approach is a highly promising one for use in computing complex flow problems.